ON THE CONVERGENCE OF THE DENSITY OF TERRAS' SET

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Abstract. The Collatz Conjecture's connection to dynamical systems opens it to a variety of techniques aimed at recurrence and density results. First, we turn to density results and strengthen the result of Terras through finding a strict rate of convergence using a recurrence argument. This rate gives a preliminary result on the Triangle Conjecture, which describes a set nodes that would dominate $L^C = \{y \in \mathbb{N} \,|\, T^k(y) > y, \forall k \in \mathbb{N}\}$. Next, we extend this argument to several situations considered in previous literature. Third, we extend prior arguments to show that the construction of several classes of measures imply the bounded trajectories piece of the Collatz Conjecture.

1. Introduction

The Collatz Map $T : \mathbb{N} \to \mathbb{N}$ given by

$$T(x) = \begin{cases} \frac{x}{2} & x \text{ even} \\ \frac{3x+1}{2} & x \text{ odd} \end{cases} \tag{1}$$

poses difficult questions on recurrence due to its complex behavior. The famous Collatz conjecture (or Syracuse conjecture, Kakutani Conjecture, Ulam's Conjecture) states that for any $n \in \mathbb{N}$, $T^k(n) = 1$ for some $k \in \mathbb{N}$, or that the Collatz map returns to the cycle $\{1, 2\}$ for all values. This may be broken into two pieces: first that every value returns to a cycle, or the bounded trajectories conjecture, and second that the only cycle is $\{1, 2\}$.

It was shown in [2] that the natural numbers may be broken into 3 components, $\mathbb{N} = C \cup D_1 \cup D_2$, where C is the set of all elements of cycles of the Collatz map, D_1 is the set of all values returning to a cycle, and D_2 is the set of all nodes which do not return to a cycle. The bounded trajectories conjecture then says $D_2 = \emptyset$ and the unique cycle conjecture states $C = \{1, 2\}$. The same paper gives a critereon for the former conjecture, that D_2 is empty if and only if there exists a finite measure μ defined on every value in $\mathbb{N}$ which is everywhere nonzero and power bounded with respect to the map T. This result was extended in [3] to a more general class of maps.

Furthermore, [3] investigated another property of the inverse Collatz map. Recall that for any point $a \in \mathbb{N}$, we may consider all preimages of a given by $\cup_{i=0}^{\infty} T^{-i}(a)$, which may be referred to as the inverse tree generated by a due to the organization of these values discussed in [3]. A further extension of this set may be considered.

Definition 1. *Let $a \in \mathbb{N}$. The Chain-Tree generated by a is $\cup_{i=0}^{\infty} \cup_{j=0}^{\infty} T^{-j}(T^i(a))$. A chain in this set is a collection of elements indexed by the integers, $\{a_z\}_{z \in \mathbb{Z}}$ where $T(a_z) = a_{z+1}$.*

The chain tree is stable under both the forward and inverse Collatz map, and any two nodes within the same chain tree will generate the same chain tree under this definition. The chain is an arbitrary choice indexing the "levels" of the chain-tree (see Figure 1), and certain chains, such as the "leftmost" chain discussed in [3] have structural properties. However, this labeling is enough to weaken the requirements on a measure to show that D_2 is nonempty. See section 4.

The same paper left an open question related to the structure of the inverse "triangle" first posed by I. Assani. Consider the partitions of $\mathbb{N}$ by mod 3 remainders $\mathbb{N}_0, \mathbb{N}_1, \mathbb{N}_2$. For $a_0 \in \mathbb{N}_0 \cup \mathbb{N}_1$, $T^{-1}(a_0) = \{2a_0\}$, and for $a_2 \in \mathbb{N}_2$, $T^{-1}(a_2) = \{2a_2, \frac{2a_2-1}{3}\}$. This offers an immediate arrangement of the inverse

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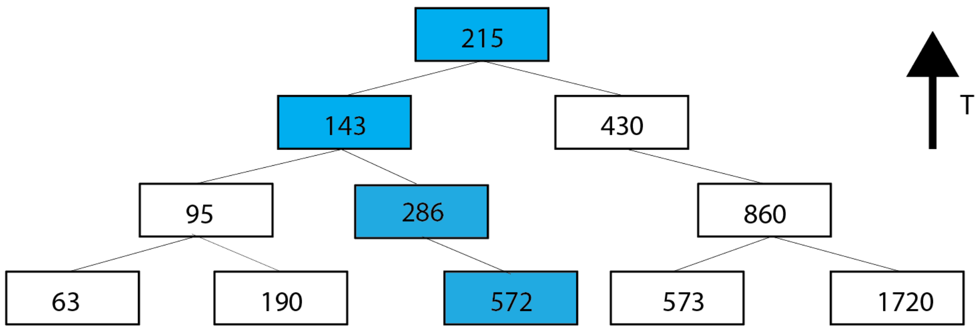

Figure 1. A Snippet of a Chain-Tree with the Chain section highlighted in Blue

images. Further, each $\mathbb{N}_2$ node a_2 is of the form $3^k h - 1$ where h is not a multiple of 3, $k \geq 1$. Using the above formula, $T^{-1}(3^k h - 1) = \{2(3^k h - 1), 3^{k-1}(2h) - 1\}$, so that there exists a sequence of k preimages $3^{k-1}(2h) - 1, 3^{k-2}(2^2 h) - 1, ..., 2^k h - 1$ which is strictly decreasing. The open conjecture states that this is the only such sequence of preimages in $\bigcup_{i=0}^{k} T^{-i}(3^k h - 1)$. We call this the Triangle Conjecture. Precisely, fix a $\mathbb{N}_2$ node $3^k h - 1$, where $k, h \geq 1$. Then, the triangle conjecture says that there exists no $a \in T^{-k}(3^k h - 1)$ such that $a \neq 2^k h - 1$ and $a \leq T^l(a)$ for all $1 \leq a \leq k$.

Recall the set $L = \{y \in \mathbb{N} \,|\, \exists k \in \mathbb{N}$ such that $T^k(y) < y\}$ of Terras [16]. The Triangle Conjecture creates a set of nodes which dominates L^c, locating possible nodes with no lesser image. The two concepts are closely intertwined.

In the first section, we strengthen the density result od Terras by finding a rate of convergence of the density of the set L to 1. This rate of convergence provides a rough upper bound on the number of possible nodes in $a \in T^{-k}(3^k h - 1)$ which have no lesser image in $\{a, T(a), ..., T^k(a)\}$, providing a partial result on the triangle conjecture.

In the second section, we use the argument and structure developed in that rate of convergence to ascribe an analogous result to a much broader class of Syracuse maps, showing that the same type of density result holds even for maps conjectured to have unbounded trajectories.

In the third section, we investigate some generalizations of the arguments of [2] and [3] to give weaker requirements for the measures involved in showing D_2 empty.

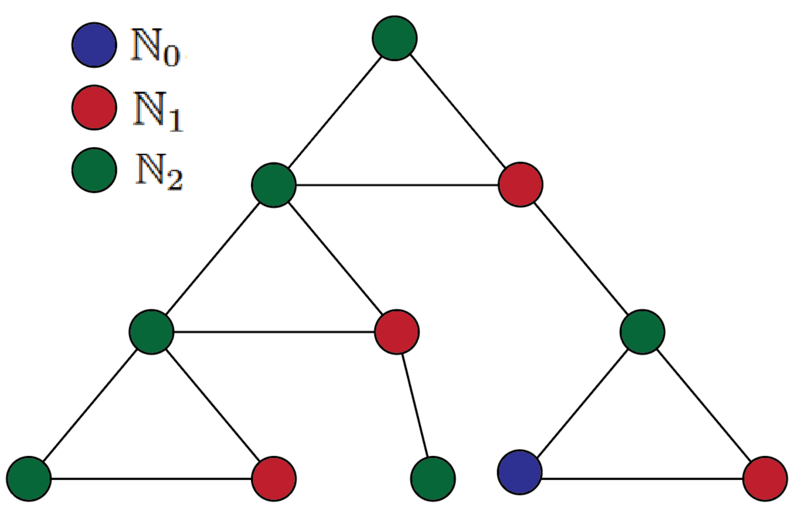

Figure 2. The Triangle for $k = 3$

2. An Introduction to Density Results

In lieu of results on strict cases, it is worthwhile to at least obtain some information on sets with full density in the natural numbers. An important foundational case for the Collatz map is the set $L = \{y \in \mathbb{N} \mid \exists k \in \mathbb{N}$ such that $T^k(y) < y\}$ introduced by R.Terras [16], which Terras proved to have density 1. This result was extended to show the set that $M_c = \{y \in \mathbb{N} \mid \exists k \in \mathbb{N}$ such that $T^k(y) < y^c\}$ has density 1 for $c \geq \log_3(2)$ by I.Korec [8]. Further analysis into these densities via statistical properties and probability distributions have been carried out authors such as Lagarias [11], Kontovorich, and Sinai ([7], [14]). Recently, T.Tao has also shown that the set D_2 has logarithmic density 1 [15] through an argument using representation theory and probability distributions. In this section, we shall focus on introducing an alternate proof of Terras' original result which gives more precisely the rate of convergence of the density of the set L to 1. First, we recall some structures original to Terras.

Definition 2. *For $k, y \in \mathbb{N}$, define $E_k(y)$ to be the vector of length k whose i^{th} component is 1 if $T^{i-1}(y)$ is odd, and 0 if it is even. Define $S_k(y)$ to be the sum of the elements of $E_k(y)$.*

Also, we use from [16] the following proposition.

Proposition 1. *$E_k(y) = E_k(x)$ if and only if $x \equiv y \mod 2^k$.*

This proposition is often called periodicity for the Collatz map, as it allows consideration of a finite number of numbers which may be extrapolated to a density. With these, we may introduce a refinement of Terras' result.

Theorem 2. *For fixed $k \in \mathbb{N}$, let $L_k = \{y \in \mathbb{N} \mid \exists\, m,\ 1 \leq m \leq k,$ such that $T^m(y) \leq y\}$. The density of L_k^C is at most*

$$\frac{2^m}{2^k} \prod_{n=0}^{m} \frac{2n+1}{n+1} \tag{2}$$

where $m = \lfloor \frac{k}{2} \rfloor$.

To prove this theorem, we will take 3 steps. First, we will introduce a general structure used in this section and the next, which has the form of a Pascal or Catalan triangle with a set of restrictions. Second, we will connect this general form to a specific triangle related to the Collatz map. Third, we will use the structure of the triangle to compute the upper bound.

Second, we will look at a specific case of this triangle and some basic results about it. Third, we will connect this to the Collatz map.

Step 1: Take a map $\tau : \mathbb{N} \to \mathbb{R}$. We define a sequence of sequences, $\{\{x_i^n\}_{i\geq 0}\}_{n\geq 0}$ where $x_0^n = 1$ for all n and for $n > 1$,

$$x_k^n = \begin{cases} x_k^n + x_{k-1}^n & k \leq \tau(n) \\ \\ 0 & \text{else} \end{cases} \tag{3}$$

Consider the n^{th} sequence to correspond to the n^{th} row of the constructed triangle. For example, if we take $\tau(k) = k$, then this defines the standard Pascal Triangle. The function τ restricts when the rows may expand to have more nonzero values in the sequence.

Consider, for example, the first 11 rows of the triangle constructed by $\tau(n) = \frac{n}{2}$ (starting at the 0^{th} row).

	$i=$0	1	2	3	4	5...		
$n=0$	1	0	0	0...				
$n=1$	1	0	0	0...				
$n=2$	1	1	0	0...				
$n=3$	1	2	0	0...				
$n=4$	1	3	2	0	0...			
$n=5$	1	4	5	0	0...			
$n=6$	1	5	9	5	0	0...		
$n=7$	1	6	14	14	0	0...		
$n=8$	1	7	20	28	14	0	0...	
$n=9$	1	8	27	48	42	0	0...	
$n=10$	1	9	35	75	90	42	0	0...
	$\vdots$							

Step 2: We prove the following lemma

Lemma 3. *Let $\tau(k) = 2k$. Then, the row sum $\sum_{i=0}^{\infty} x_i^n$ gives an upper bound for the number of vectors $S_k(x)$ such that $T^m(x) > x$ for all $1 \le m \le k$.*

Proof. Consider $y \in L_k^C$. Then, we have that $\frac{T^n(y)}{y} > 1$ for all $1 \le n \le k$. Set $l = S_n(y)$ and and in particular that for all such n

$$(\frac{3y+1}{2y})^l(\frac{y}{2y})^{n-l} \ge \frac{T^n(y)}{T^{n-1}(y)}\frac{T^{n-1}(y)}{T^{k-2}(y)}...\frac{T(y)}{y} = \frac{T^n(y)}{y} \ge 1 \tag{4}$$

By taking logs, we note that

$$S_n(y) \ge \frac{n\ln(2)}{\ln(3+\frac{1}{y})} \tag{5}$$

so that $S_n(y) \ge \frac{n}{2}$ in general since $y \ge 1$. Therefore, the number of vectors satisfying this inequality for $n \le k$ bounds the number of vectors so $T^n(y) > y$ for $n \le k$.

To count the number of $E_k(x)$ vectors satisfying this restriction, we may construct a recurrence relation by counting the number of 0s possible in the vector. There is always only a single vector with no 0s, the vector of all 1s. If the number of 0s, l, is less than $\frac{n}{2}$, then when considering a E_{n+1} vector, we may take a vector with l zeroes and attach another 0 or add a 1 at the end while still satisfying this inequality. Therefore, the number of E_{n+1} vectors with l zeroes, $1 \le l \le \frac{n}{2}$, is the number of E_n vectors with l zeroes plus the number with $l-1$ zeroes. Now, consider that if $l > \frac{n}{2}$ and if $l \le \frac{n+1}{2}$, the number of E_{n+1} vectors with l zeroes is precisely the number of E_n vectors with $l-1$. For all other values, there are 0 vectors satisfying the relations. Tracing these recurrence relations shows that the number of E_n vectors with i zeroes is precisely x_i^n for $\tau(k) = 2k$. Therefore, counting these vectors reduces to taking the row sum of the triangle constructed by this τ. Fix the $\{\{x_i^n\}_{i\ge0}\}_{n\ge0}$ as those generated this way. □

Notice that that row sums are strictly increasing, so it suffices to consider only the odd-number rows to generate an upper bound. From the triangle above, this amounts to considering the rows

$$\begin{array}{llllll} n=1 & 1 \\ n=3 & 1 & 2 \\ n=5 & 1 & 4 & 5 \\ n=7 & 1 & 6 & 14 & 14 \\ n=9 & 1 & 8 & 27 & 48 & 42 \\ & & \vdots \end{array}$$

Consider now this triangle within its own right. Define, now, a sequence of sequences for this triangle $\{\{y_i^n\}_{i\geq 0}\}_{n\geq 0}$ where $y_i^n = x_i^{2n+1}$. Considering the recurrence relation row-wise, $y_k^n = y_{k-2}^{n-2} + 2y_{k-1}^{n-2} + y_k^{n-2}$.

This triangle develops a more Catalan-like relation, and this is not the first time it has appeared. In particular, Shapiro [13] showed that for the largest index k so that $y_k^n \neq 0$, y_k^n is precisely the n^{th} Catalan number, $\frac{1}{n+1}\binom{2n}{n}$. He also computed the row sums of this triangle, which we repeat in a simplified way for posterity.

Lemma 4. *The row sum of the triangle on x_i^n at level $2n+1$ is*

$$\sum_i x_i^{2n+1} = \sum_i y_i^n = 2^n \prod_{k=0}^{n} \frac{2k+1}{k+1} \tag{6}$$

Proof. We apply induction to the triangle $\{\{y_k^n\}\}$, noting that the n-th level of this triangle corresponds to the $2n+1$-st level of the $\{\{x_k^n\}\}$ triangle.

The formula given is immediate in the case $n=0$ or $n=1$ from the values computed above. Let it be shown for values up to $n-1$ and consider row n. Then, also note that

$$\sum_i y_i^n = \sum_i y_{i-2}^{n-1} + 2y_{i-1}^{n-1} + y_i^{n-1} = 4(\sum_i y_i^{n-1}) - y_{\frac{n-1}{2}}^{n-1} = 4(\sum_i y_i^{n-1}) - y_k^{n-1} \tag{7}$$

where $\frac{n-1}{2} = k$ is the largest index i so $y_i^{n-1} = x_i^{2n-1}$ is nonzero. Now, we may apply the inductive assumption and the result from [13] to note that this sum is

$$4(2^{n-1}\prod_{k=0}^{n-1}\frac{2k+1}{k+1}) - \frac{1}{n+1}\binom{2n}{n} \tag{8}$$

With some algebraic manipulation,

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\frac{(2n)!}{n!n!}\right) = \frac{1}{n+1}\left((\frac{2}{1}\times\frac{4}{2}\times ... \times\frac{2n}{n})(\frac{1}{1}\times\frac{3}{2}\times ... \times\frac{2n-1}{n})\right) \tag{9}$$

$$= \frac{2}{n+1}\left(2^{n-1}\prod_{k=0}^{n-1}\frac{2k+1}{k+1}\right) \tag{10}$$

□

Step 3: We now prove the theorem.

Proof. The number of unique $E_k(x)$ vectors is 2^k by Terras' periodicity. The number of such vectors satisfying $S_n(x) > \frac{x}{2}$ for all $n \leq k$ is then bounded above by

$$2^m \prod_{n=0}^{m} \frac{2n+1}{n+1} \tag{11}$$

for $m = \lfloor \frac{k}{2} \rfloor$ by lemma 4 and lemma 3. Thus, the density of L_k^C is then at most

$$2^{m-k} \prod_{n=0}^{m} \frac{2n+1}{n+1} \tag{12}$$

□

Corollary 1. *The density of L is 1.*

Proof. Since $L^C \subset L_k^C$ for all k, we have that the density of L^C is at most

$$\lim_{k\to\infty} \frac{2^m}{2^k} \prod_{n=0}^{m} \frac{2n+1}{n+1} = \lim_{k\to\infty} \frac{2^m}{2^{k-m-1}} \prod_{n=0}^{m} \frac{2n+1}{2n+2} \leq \lim_{m\to\infty} 2(\prod_{n=0}^{m} \frac{2n+1}{2n+2}) \tag{13}$$

for m a function of k as above. Note then that the rightmost limit is then

$$\exp(\lim_{m\to\infty} \ln(2) + \sum_{n=0}^{m} \ln(1 - \frac{1}{2n+2}) \tag{14}$$

for $x < 1$, we have that $ln(1-x) \leq -x$ since $f(x) = \ln(1-x) + x$ has $f(0) = 0$ and $f'(x) = \frac{-x}{1-x} \leq 0$. Thus, since the exponential is an increasing function, this limit is at most

$$\exp(\lim_{m\to\infty} \ln(2) + \sum_{n=0}^{m} -\frac{1}{2n+2}) \tag{15}$$

The series is harmonic and thus diverges to negative infinity, so that the total limit is 0. □

2.1. **Bounding the Number of Failures within the Triangle.** Recall from [3] the conjecture on the inverse image triangle of a $\mathbb{N}_2$ node. Precisely, it is conjectured that given any $k, n \in \mathbb{N}$, $a \in T^{-k}(3^k h - 1)$, $a \neq 2^k h - 1$, then there exists some m so $1 \leq m \leq k$ such that $T^k(a) < a$. Since we may consider $3^{k-1}(3h) - 1$ and further reductions, this is to say that this holds for all nodes in the preimages $\bigcup_{i=1}^{k} T^{-i}(3^k h - 1)$ not of the form $2^a 3^{k-a} h - 1$, i.e. the leftmost branch as distinguished in that paper as well.

This result would locate the values not in L as defined in section 3, by generating a set in which, if values in L^c exist, they must be located. The main theorem of the prior section introduces an upper bound for the number of such exception cases.

Corollary 2. *For a fixed $k, h \in \mathbb{N}$, the number of $a \in T^{-k}(3^k h - 1)$ such that for all m so $1 \leq m \leq k$ has $T^m(a) > a$ is at most*

$$2^m \prod_{n=0}^{m} \frac{2n+1}{n+1} \tag{16}$$

where $m = \lfloor \frac{k}{2} \rfloor$.

Proof. The number of nodes a in the triangle generated by $3^k h - 1$ such that $T^m(a) > a$ for $1 \leq m \leq k$ is at most the number of $E_k(y)$ vectors corresponding to $1 \leq y \leq 2^k$ so $T^m(y) > y$ for $1 \leq m \leq k$. Thus, this is given precisely as in lemma 3 and lemma 4. □

We may also simplify the number of cases to calculate drastically by examining the structure of the triangle and how it varies across different values of h.

Proposition 5. *The structure of the triangle generated by $3^k h - 1$ is invariant with respect to h. That is to say, for each $a \in T^{-l}(3^k h - 1)$, $1 \leq l \leq k$, and for any $h_1 \in \mathbb{N}$, there exists $a_1 \in T^{-l}(3^k h_1 - 1)$ such that $E_l(a) = E_l(a_1)$.*

Proof. Fix $k, h \in \mathbb{N}$ and consider $a \in T^{-l}(3^k h - 1)$ for $1 \leq l \leq k$. We wish to show that there exist α, β not dependent on h so that $a = 2^l 3^{k-l} h\alpha + \beta$. This decomposes a into the part maintaining the initial power of 3 ($2^l 3^{k-l}\alpha$) and the "remainder" part that helps locate it on the branch β. Indeed, consider that the preimage of $3^k h - 1$ is $\{2(3^{k-1})h - 1, 2(3^k h) - 2\}$. Working inductively, if we assume there are α_0 and β_0 not dependent on h so $T(a) = 2^{l-1}3^{k-l+1}h\alpha_0 + \beta_0$, then $a \in \{2^l 3^{k-l} h\alpha_0 + \frac{2\beta_0 - 1}{3}, 2^l 3^{k-l} h(3\alpha_0) + 2\beta_0\}$ where, when the first option is possible, $\frac{2\beta_0-1}{3}$ is an integer. Therefore, we may express a in the desired form as well.

Now, consider $E_l(a)$. Denote, for arbitrary $h_1 \in \mathbb{N}$, $a_1 = 2^l 3^{k-l} h_1 \alpha + \beta$. By Terras' periodicity result, $E_l(a_1) = E_l(a)$, and further this implies by repeated applications of the Collatz map that $T^l(a_1) = 3^k h_1 - 1$. □

Proposition 6. *For a fixed k, the value $a \in T^{-k}(3^k h - 1)$ has an associated node a_1 so $T^m(a_1) > a_1$ for all $1 \leq m \leq k$ if and only if*

$$3^{S_m(a)} > 2^m \tag{17}$$

for the same m

Proof. In the case $k = 1$, $\frac{T(a)}{a}$ is either $1/2$ or $\frac{3}{2} - \frac{1}{2(2^k h\alpha+\beta)} < \frac{3}{2}$. Therefore, in the case that $a < T^m(a)$

$$\frac{T^m(a)}{a} = \frac{T^m(a)}{T^{m-1}(a)}\frac{T^{m-1}(a)}{T^{m-2}(a)} \cdots \frac{T^2(a)}{T(a)}\frac{T(a)}{a} < (\frac{1}{2})^{m-S_m(a)}(\frac{3}{2})^{S_m(a)} = \frac{3^{S_m(a)}}{2^m} \tag{18}$$

Consider now that $3^{S_m(a)} > 2^m$ for all such $m \leq k$. Then, we may take some $\alpha = \min\{a, T(a), ..., T^k(a)\}$ and the same computation above gives

$$\frac{T^m(a)}{a} > 2^{-m}\left(3 - \frac{1}{\alpha}\right)^{S_m(a)} \tag{19}$$

Since $\alpha \to \infty$ as $h \to \infty$, for sufficiently large h, we then have that the right is greater than 1 by the assumption. Denote such an h as h_1 and the associated point on the tree (as in the previous proposition) as a_1, finishing the proof. □

Proposition 11 reduces solving the triangle conjecture to looking at a limiting ratio between the values in the triangle and the top of the triangle. This gives a more rigid structure to check across k without worrying about h.

Further, the invariance with respect to h occurs beyond $h \in \mathbb{N}$, and considering an extension of the Collatz map to $T : \mathbb{Z} \to \mathbb{Z}$ allows consideration of $h = 0$ or to looking at the triangle of k levels generated by -1, which has the same structure as for $h = 1$ well.It is not yet clear how this extension connects to the triangle conjecture, but is an interesting point nonetheless.

3. More General Delayed-Phase Triangles

3.1. **Considerations on Syracuse Maps.** We now extend our considerations from the Collatz map to some more general maps of the form in [3]

$$V(x) = \begin{cases} \frac{m_i x + r_i}{d} & x \equiv i \mod d \end{cases} \tag{20}$$

where $r_i = -im_i \mod d$ and $m_0 m_1 ... m_{d-1}$ is relatively prime to d. This definition was referenced from [12] and used also in [3]. We will not require the relatively prime assumption for this work.

We take the additional assumption that $m_i < d$ for some i. Without this, the maps would grow in almost all cases and their analysis wouldn't be worthwhile.

Next, define $L^V = \{x \in \mathbb{N} \mid V^k(x) < x \text{ for some } k \in \mathbb{N}\}$. We wish to show that the density of the set L^v is 1. We do this through the simplified sets $L^V_k = \{x \in \mathbb{N} \mid V^m(x) < x \text{ for some } m \leq k\}$.

To begin with, consider a vector $v \in \{0,1\}^n$, and we define $S_k(v)$ to be the number of ones appearing in the first k components of v. We wish to count the number of vectors in $\{0,1\}^n$ satisfying $S_k(v) \geq \frac{k}{\alpha}$ for some $\alpha \in \mathbb{Z}$ so $\alpha \geq 2$ and $1 \leq k \leq n$.

To do this, consider a delayed-phase Pascal's triangle with $\tau(k) = (1 - \frac{1}{\alpha})k$. This means that we take a sequence of sequences $\{\{x^n_k\}_{k\geq 0}\}_{n\geq 0}$ where we define $x^0_n = 1$ for all n and define all other values by the relationship

$$x^n_k = \begin{cases} x^{n-1}_{k-1} + x^{n-1}_k & k \leq \lfloor n(1 - \frac{1}{\alpha})\rfloor \\ 0 & \text{else} \end{cases}$$

In more direct words, this triangle has a "skip" every α rows. For example, $\alpha = 3$ looks like:

	$i =$0	1	2	3	4	5...
$n = 0$	1	0	0	0...		
$n = 1$	1	0	0	0...		
$n = 2$	1	1	0	0...		
$n = 3$	1	2	1	0...		
$n = 4$	1	3	3	0	0...	
$n = 5$	1	4	6	3	0...	
$n = 6$	1	5	10	9	3	0...
$n = 7$	1	6	15	19	12	0...

Note that x_k^n denotes the number of vectors of length n with k zeroes such that $S_k(v) \geq \frac{k}{\alpha}$ for all $k \leq n$.

Proposition 7. *Denote by $R_\alpha(n)$ the balanced row-sum $2^{-n}\sum_k x_k^n$ for the recurrence triangle generated by this $\alpha \geq 2$. Then,*

$$\lim_{n\to\infty} R_n = 2 - C^\alpha(2^{-\alpha}) \tag{21}$$

where $C^\alpha(x)$ is the generating function for the Catalan-Fuss numbers.

Proof. First, note that the values x_k^n are increasing with α, so that if we show this holds for large α, the smaller cases follow immediately.

Second note that this ratio R_n is constant when rows of the triangle do not "skip", or when the growth of k is not restricted. Therefore, we only analyze the levels in which these R_n change, and to do so we define a new triangle by

$$y_j^i = x_{(\alpha-1)i-j}^{\alpha i+1} \tag{22}$$

Since the x_k^n satisfy the relationship $x_k^n = \sum_{l=0}^{\alpha}\binom{\alpha}{l}x_{k-l}^{n-\alpha}$ when the x_k^n is nonzero, we can translate the same relationship to the y_j^i:

$$y_j^i = \sum_{k=0}^{n}\binom{n}{k}y_{(j+1)-k}^{i-1} \tag{23}$$

where we consider $y_j^i = 0$ for $i < 0$. This characterization is not strictly necessary, but it emphasizes the important traits of the triangle as in the prior case.

Define $S_n = \sum_{j=1}^{\infty} y_j^n$ and note that $S_n = 2^\alpha S_{n-1} - y_0^{n-1}$. Therefore, since $S_0 = 1$ we have that $S_n = 2^{\alpha n} - \sum_{i=0}^{n-1} 2^{\alpha(n-1-i)}y_0^i$ and $R_{n\alpha+1} = 1 - \sum_{i=0}^{n-1} 2^{-(i+1)\alpha}y_0^i$. The above observation on the R_n then says that $R_{n\alpha+1} = R_{n\alpha+2} = ... = R_{n\alpha+1+\alpha-1} = 1 - \sum_{i=1}^{n-1} 2^{-i\alpha}y_0^i$.

Let us now define a k-Dyck path. A k-Dyck path of length $(k+1)n$ is a series of $(k+1)n$ up or down steps. Up steps move right 1 and up k, while down steps move down 1 and right 1. In particular, we are Moving from the point $(0,0)$ in the plane to the point $((k+1)n, 0)$ while keeping y-values non-negative. In some cases, this may also be called an up-down path. We may denote a path by a vector of 1s and 0s, where we have a 1 denoting an up step and a 0 denoting a down step.

Consider that an $(\alpha-1)$-Dyck path of length $(\alpha)n$ must have also precisely $(\alpha-1)n$ down steps, or 0s. Furthermore, the restriction that the path keeps in non-negative y is precisely the restriction that the sum over the first k values of the vector must be at least $\frac{k}{\alpha}$, corresponding to $S_k(v) \geq \frac{k}{\alpha}$. This is to say that the number of $(\alpha-1)$-Dyck paths of length αn is precisely $x_{(\alpha-1)n}^{\alpha n} = y_0^{n-1}$ (since $x_{(\alpha-1)n}^{\alpha n} = x_{(\alpha-1)(n-1)}^{\alpha(n-1)+1}$ directly from the recurrence).

Thus, we have that for $n > 0$, from the second page of [1],

$$y_0^n = \frac{1}{(\alpha-1)(n+1)+1}\binom{\alpha(n+1)}{(n+1)} = \frac{1}{\alpha(n+1)+1}\binom{\alpha(n+1)+1}{n+1}$$

Other values of the y_k^n relate to down-step statistics of Dyck paths investigated in [1] as well.

Since these are the standard Fuss-Catalan numbers, we may use the generating function

$$C^{\alpha}(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{\alpha n+1}\binom{\alpha n+1}{n} x^n = 1 + \sum_{n=0}^{\infty} y_0^n x^{n+1}$$

and finally

$$\lim_{n\to\infty} 2 - R_n = \lim_{n\to\infty} 2 - R_{n\alpha+1} = C^{\alpha}(2^{-\alpha}) \tag{24}$$

This establishes the claim. □

Lemma 8. *For $\alpha > 2$, we have $C^{\alpha}(2^{-\alpha}) = \sum_{n=0}^{\infty} 2^{-\alpha n}\left(\frac{1}{\alpha n+1}\binom{\alpha n+1}{n}\right) < 2$ is the unique root of $y^{\alpha} - 2^{\alpha}y + 2^{\alpha}$ in $(1,2)$.*

Proof. It is a common relation on this generating function ([5] page 347) that

$$x(C^{\alpha}(x))^{\alpha} = C^{\alpha}(x) - 1$$

so that $C^{\alpha}(2^{-\alpha})$ is a root of $y^{\alpha} - 2^{\alpha}y + 2^{\alpha}$ greater than 1. Since the derivative of this polynomial, $\alpha y^{\alpha-1} - 2^{\alpha}$, has at most 2 roots and these are of the same magnitude in $(1,2)$, and since there is a root of the original polynomial at 2, there is a single root in $(1,2)$.

By the ratio test, $C^{\alpha}(2^{-\alpha})$ converges for all α. Furthermore, for $n > 2$

$$\frac{1}{\alpha n+1}\binom{\alpha n+1}{n} < 2^{\alpha-2} \tag{25}$$

we recover that $C^{\alpha}(2^{-\alpha}) < 2$.

□

Using this lemma and the previous proposition, we may almost recover a result of H.Möller in the case $d = 2$:

Corollary 3. *Any Syracuse map*

$$V(x) = \begin{cases} \frac{tx+1}{2} & x \text{ odd} \\ \frac{x}{2} & x \text{ even} \end{cases} \tag{26}$$

for $t \geq 7$ has L^V of density less than 1.

Proof. For $y \in (L^V)^C$

$$\frac{V^m(y)}{y} \geq \frac{1}{2}^m (t)^{S_m(y)} \tag{27}$$

such that the number of vectors so

$$S_m(y) \geq m\frac{\ln(2)}{\ln(t)} \tag{28}$$

is bounded above by the number of $y \in (L_m^V)^c$ by the extended periodicity. Since $t \geq 7$, we have that this is at least the number of vectors so $S_m(v) \geq \frac{m}{3}$, so the previous proposition and lemma show $\lim_{n\to\infty} R_n > 0$.
□

The case of the $5x + 1$ map simply requires a different consideration of this connection to Dyck paths, which will be explored later.

3.2. **Powers of x.** We now look at the case of $M_c = \{x \in \mathbb{N} \,|\, T^k(x) < x^c \text{ for some } k \in \mathbb{N}\}$ for $c \in (0,1)$ and how that corresponds to the same triangles. Using the same string of calculations, $y \notin M_c$ implies

$$T^m(y) > y^c \Rightarrow S_m(y) \geq \frac{m\ln(2) + ln(y)(c-1)}{ln(3+1/\beta)} \tag{29}$$

For sufficient β as in the previous cases. We consider the case $\beta = 1$ to allow for computation with Dyck paths, though this may perhaps be strengthened. Then, the number of $y \leq 2^k$ so the above holds for all $m \leq k$ is bounded above by the number of vectors so

$$S_m(v) \geq \frac{m}{2} - (1-c)\frac{k}{2} \tag{30}$$

for all $m \leq k$. We may generate the delayed-phase triangle with the rule $\tau(n) = \frac{n}{2} + (1-c)\frac{k}{2}$. For example, if we set $c = 1/2$ and $k = 8$, we see the triangle

	$i=$0	1	2	3	4	5...	
$n = 0$	1	0	0	0...			
$n = 1$	1	1	0	0...			
$n = 2$	1	2	1	0...			
$n = 3$	1	3	3	1	0...		
$n = 4$	1	4	6	4	1	0...	
$n = 5$	1	5	10	10	5	0...	
$n = 6$	1	6	15	20	15	5	0...
$n = 7$	1	7	21	35	35	20	0...

Now, we may construct consider the levels in which "skips" occur exactly as before to generate a new triangle $\{\{y_k^n\}\}$. Then,

$$y_k^n = x^{\lfloor (1-c)k \rfloor - 1 + 2n} \tag{31}$$

related by $y_k^n = y_{k-1}^{n-1} + 2y_k^{n-1} + y_{k+1}^{n-1}$, where y_k^{-1} is considered to be 0 for all k. In particular, this triangle $\{\{y_k^n\}\}$ starts with $\{y_k^0\}_k$ a row of the binomial triangle, then proceeds recurrently the same way as in the proof for the Collatz map without power.

Lemma 9. *For $m = \lfloor (1-c)k \rfloor - 1$,*

$$y_0^n = \frac{m+2}{2n+m+2}\binom{2n+m+2}{n} \tag{32}$$

Proof. We consider the same Dyck-path connection as in the case of the Syracuse maps. Note that $y_k^n = x_{m+n}^{m+2n} = x_{m+n+1}^{m+2n+1}$ is the vector of length $2n+m+1$ satisfying $S_k(v) \geq \frac{k}{2} - m$ for all $k \leq m+n+1$ with $m+n+1$ 1s.

This is to say that it is the same as a Dyck path from $(0, m+1)$ to $(2n+m+1, 0)$ with up-steps $(1,1)$ and down-steps $(1,-1)$ staying above the x-axis. In particular, y_0^n represents the number of Dyck paths from $(0, m+1)$ to $(2n+m+1, 0)$ with up-steps $(1,1)$ and down-steps $(1,-1)$.

By corollary 2.4 in [4], the number of Dyck-paths from $(0,0)$ to $(2n+m+1, m+1)$ with up-steps $(1,1)$ and down-steps $(1,-1)$ is

$$\frac{m+2}{2n+m+2}\binom{2n+m+2}{n} \tag{33}$$

by symmetry, this is precisely the same as the number from $(0, m+1)$ to $(2n+m+1, 0)$, and so this gives y_0^n. □

In fact, using the same argument gives

Corollary 4. *For the triangle generated by $\tau(n) = (1 - \frac{1}{\alpha})n - m$, we have that*

$$x_{n+m}^{\alpha n+m} = y_0^n = \frac{m+2}{\alpha n + m + 2}\binom{\alpha n + m + 2}{n} \tag{34}$$

However, we will continue in the $\alpha = 2$ case of the Collatz map, due to the fact that the binomial coefficients $\frac{m+2}{2n+m+2}\binom{2n+m+2}{n}$ have a well-understood generating function. In fact, these coefficients are precisely convolutions of Catalan numbers. Thus, we may reach a clean bound for the row sums.

Proposition 10. *The n-th row sum of the triangle generated by $\tau(n) = \frac{n}{2} + (1-c)\frac{k}{2}$ is*

$$\begin{cases} 2^n & n \leq m+1 \\ 2^n - \sum_{k=0}^{\frac{n-m}{2}-1}\binom{n+1}{k} & \frac{n-m}{2} \in \mathbb{N},\ n > m+1 \\ 2(2^{n-1} - \sum_{k=0}^{\frac{n-m-1}{2}-1}\binom{n}{k} & \frac{n-m-1}{2} \in \mathbb{N},\ n > m+1 \end{cases} \tag{35}$$

where $m = \lfloor (1-c)k \rfloor - 1$.

Proof. First, by the Catalan Convlution formula, the nth number of the kth Catalan convolution is $\frac{k}{2n+k}\binom{2n+k}{n}$. For $k = m+2$, this is precisely the y_0^n computed above.

Next, construct a generating function for the row sums. Let the sum at the nth level of the triangle $\{\{y_k^n\}\}$ starting with the m-th binomial row be $f_m(n)$. Define $F_m(x) = \sum_{n=0}^{\infty} f_m(x)x^n$. We know that $f_m(0) = 2^m$. Therefore, we may use the recursion definition of the sum to obtain

$$F_m(x) = 2^m + \sum_{n=1}^{\infty}(4f_m(n-1) - c_{m+2}(n-1))x^n$$

where we denote by the $c_k(n)$ the n-th number in the k-th convolution of Catalan numbers. Note that the Catalan numbers have generating function $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, so that the k-th convolution has generating function $C(x)^k$. Then, the above simplifies to

$$F_m(x) = \frac{1}{1-4x}(2^m - xC(x)^{m+2}) = 2^m\left(\sum_{n=0}^{\infty}(4x)^n\right) - \frac{xC(x)^m}{1-4x}$$

Applying the fact that

$$\frac{C(x)^k}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\binom{2n+k}{n}x^n$$
$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n$$

we may realize that

$$x\frac{C(x)^{m+2}}{1-4x} = \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n-1}\binom{2k}{k}\binom{2(n-k)+m+2}{(n-k)}$$

This may computed with integral representations, called the Egorychev method. In our case, we use only the fact that

$$\binom{n}{k} = res_0(\frac{(1+z)^n}{z^{k+1}})$$

to write that

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{2k}{k}\binom{2(n-1-k)+m+2}{(n-1-k)} &= \sum_{k=0}^{\infty}\binom{2k}{k}\binom{2(n-1-k)+m+2}{(n-1-k)}\\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+m-2k}}{z^{n-k}}\sum_{k=0}^{\infty}\binom{2k}{k}dz\\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+m}}{z^{n}}\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{z}{(1+z)^2}\right)dz\\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+m}}{z^{n}}\frac{1}{\sqrt{1-4(\frac{z}{(1+z)^2}}}dz\\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+m}}{z^{n}}\frac{1+z}{\sqrt{(1-z)^2}}dz\\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+m+1}}{z^{n}}\frac{1}{(1-z)}dz\\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{2n+m+1}}{z^{n}}\sum_{k=0}^{\infty}z^k dz\\
&= \sum_{k=0}^{\infty}\binom{2n+m+1}{n-1-k}\\
&\sum_{k=0}^{n-1}\binom{2n+m+1}{k}\end{aligned}$$

Putting this back into the original sum gives that the row sum $\sum_{i=0}^{\infty} y_i^n$ is

$$2^{m+2n}-\sum_{k=0}^{n-1}\binom{2n+m+1}{k}$$

Note that the $\{\{x_k^n\}\}$ triangle gives powers of two up until the $(m+1)$st row, then doubles on each row a "skip" does not occur. Hence, translating this to the form given in the proposition finishes the proof. □

3.3. **A Connection Between the Triangle and Lattice Paths.** Both of the above triangles were devoted to counting the number of vectors $v \in \{0,1\}^k$ so

$$S_m(v) \geq m\alpha - \beta \tag{36}$$

for $\alpha \in (0,1)$

In the first case, we made a connection to a triangle studied directly by Shapiro, but the connection of the second case to Dyck paths is more generally applicable. In the above case, we used the up-down step model. Let us now convert to the model of Dyck paths as lattice paths under a diagonal (i.e. paths from $(0,0)$ to some point weakly below $y = ax + b$ along lattice $\mathbb{Z}^2$ taking steps either $(x,y) \to (x+1,y)$ or $(x,y) \to (x,y+1)$ only).

For each lattice path, we may codify a horizontal step as a 1 and a vertical step as a 0 to turn the path into a binary vector v. Then, $S_m(v) \geq m\alpha - \beta$ becomes

$$x \geq (x+y)\alpha - \beta \tag{37}$$

or

$$y \leq \left(\frac{1}{\alpha} - 1\right)x + \frac{\beta}{\alpha} \tag{38}$$

thus, the number of v of length k satisfying equation 36 is the number of lattice paths satisfying inequality 38 at each step such that $x + y = k$. In particular, the y value of the endpoint counts the number of 0s in the vector. Given rational values such as these, the

Let us translate this back to the delayed-phase triangle. Generating a delayed-phase triangle from equation 36 gives that $x_k^n = x_k^{n-1} + x_{k-1}^{n-1}$ for $k \leq n(1-\alpha)+\beta$. This is the same as $\tau(n) = n(1-\alpha)+\beta$ which is to say precisely the following

Theorem 11. *Let $\{\{x_k^n\}\}$ be the delayed-phase triangle generated by $\tau(n) = n(1-\alpha)+\beta$. Then, x_k^n is the number of lattice paths from $(0,0)$ to $(n-k,k)$ weakly below $y = \left(\frac{1}{\alpha}-1\right)x + \frac{\beta}{\alpha}$.*

In general, this is not an easy number to compute either. Progress on these lattice paths has tended to focus on α and β rational, but these may still be difficult computationally depending on α and β. In the case of irrational values, we may still compute the number of paths to a fixed endpoint using a line of sufficiently close rational approximations of α and β, so that some information is known. In a simplified case, we may still obtain a closed formula:

Corollary 5. *Let $\{\{x_k^n\}\}$ be the delayed-phase triangle generated by $\tau(n) = n(1-\alpha)+\beta$. Let $m_1 = \lceil \frac{\alpha}{1-\alpha} \rceil$ and $m_2 = \lfloor \frac{\beta}{1-\alpha} \rfloor$. Then,*

$$x_k^n \leq \binom{n+m_2}{n-k} - \sum_{i=1}^{k-m_2} \binom{i(m_1+1)+m_2-1}{i+m_2} \frac{n-(m_1+1)k+m_2m_2+1}{n-m_2-i(m_1+1)+1} \binom{n-m_2-i(m_1+1)+1}{k-m_2-i} \tag{39}$$

Proof. The number of lattice paths from $(0,0)$ to $(n-k,k)$ weakly below $y = \left(\frac{1}{\alpha}-1\right)x + \frac{\beta}{\alpha}$ is strictly less than the number from $(0,-m_2)$ to $(n-k, n-k-m_2)$ weakly below the line $y = \left(\frac{1}{\alpha}-1\right)x$. This is again bounded above by the number of steps from $(0,-m_2)$ to $(n-k, n-k-m_2)$ weakly below the line $y = \frac{1}{m_1}x$. From [9] Theorem 10.4.7, we obtain the result. $\square$

While the above formula looks intimidating, it mostly encapsulates the same two arguments presented using Dyck paths in the previous sections. Indeed, using the cases described in those sections, this simplifies to the aforementioned formulas.

The same survey [9] holds some more general results on lattice paths for more complex computations.

3.4. **Higher Dimensions.** We now want to consider Syracuse maps for which the denominator d is greater than two. In this case, we now have $E_k(x) \in \{0,1,...,d-1\}^k$. In the case $d=2$, it was simple enough to count the number of 0s in a vector and so we obtained a structure that fit into a 2-dimensional lattice. In the case of more general d, this moves to a d-dimensional object. With some concessions, we may again get a formula

Proposition 12. *Let V be of the form*

$$V(x) = \begin{cases} \frac{m_i x + r_i}{d} & x \equiv i \mod d \end{cases} \tag{40}$$

Take the additional assumptions that $m_i < d$ for $i \neq j$ and $m_j > d$, and $m_i + r_i > 0$ for all i. Let $S_k^i(x)$ denote the number of values in $(x, V(x), V^2(x), ..., V^{k-1}(x))$ which are equivalent to $i \mod d$. Finally, let $A_k = \{(c_1, c_2, ..., c_d \mid \sum_i c_i = k,\ c_j \geq \sum_{i\neq j} \lfloor \frac{\ln(m_i+r_i)-\ln(d)}{\ln(m_j+r_j)} \rfloor c_i\}$

Then, the number of values $x \leq d^k$ with stopping time greater than k is bounded above by

$$\sum_{c \in A_k} \frac{c_j - \sum_{i\neq j} \lfloor \frac{\ln(d)-\ln(m_i+r_i)}{\ln(m_j+r_j)-\ln(d)} \rfloor c_i + 1}{1+k} \binom{1+k}{c_j+1, c_2, ..., c_d} \tag{41}$$

Proof. For $x \in (L^V)^C$, we have

$$\frac{1}{d}^k \prod_{i=0}^{d-1} \left(m_i + \frac{r_i}{\beta}\right)^{S_k^i(x)} > 1 \Rightarrow \tag{42}$$

$$\sum_{i=0}^{d} S_k^i(x) \ln(m_i + \frac{r_i}{\beta}) > k \ln(d) \tag{43}$$

We then apply the assumption that $m_j > d$ and all others are less than d to derive

$$\sum_{i\neq j} S_k^i(x)\left[\frac{\ln(d)-\ln(m_i+\frac{r_i}{\beta})}{\ln(m_j+\frac{r_j}{\beta})-\ln(d)}\right]\leq S_k^j(x) \tag{44}$$

Equating a step in the x_i direction to a value of i in the vector $E_k(x)$ gives that the number of vectors satisfying the above is the number of lattice paths starting at $(0,0)$ of length k such that

$$\sum_{i\neq j} x_i\left[\frac{\ln(d)-\ln(m_i+\frac{r_i}{\beta})}{\ln(m_j+\frac{r_j}{\beta})-\ln(d)}\right]\leq x_j \tag{45}$$

In particular, this number is bounded above by the number of lattice paths satisfying

$$\sum_{i\neq j} x_i\lfloor\frac{\ln(d)-\ln(m_i+\frac{r_i}{\beta})}{\ln(m_j+\frac{r_j}{\beta})-\ln(d)}\rfloor\leq x_j \tag{46}$$

so that applying theorem 10.16.1 in [9], setting $\beta = 1$, and summing across the possibilities gives the result. □

4. Measure Equivalences

In [2], it was shown that bounding the Collatz Trajectories is equivalent to constructing a measure which is finite, power-bounded with respect to the Collatz map, and everywhere nonzero. The construction was extended further to general maps including at least a single cycle in [3]. Within the case of the Collatz map, there arise difficulties in the construction of such a measure due to family-chain connections also discussed in [3] which allow values of preimages of a given point to vary widely in a hard-to-predict way. In this section, we develop further the theory surrounding such applications of measures to bounding the trajectories of the Collatz Map by extending to the cases of weaker measures, and further use a tool from dynamical systems to also gain information on the length of cycles possible depending on current knowledge of the minimum bound of cycle elements for cycles other than $\{1, 2\}$.

4.1. **Limiting Measure Cases.** The principle desire in extending the measures is to weaken the requirement on bounding the measure of given sets, thus extending the measures to take advantage of properties noticed in [2]. The following two results show that we may trade these requirements for behaviors over time of the measure, perhaps allowing for leveraging long-term decreases or increases of the values of preimages. For this section, assume each measure is non-negative and defined on the σ-algebra $P(\mathbb{N})$. The following establishes the idea behind this extension.

Proposition 13. *Let there exist a finite measure μ on $\mathbb{N}$ such that $\lim_{n\to\infty}\mu(T^{-n}(A))$ exists for all $A \subset \mathbb{N}$. Then, D_2 must have measure 0.*

Proof. By contradiction, let $a \in D_2$ have $\mu(a) > 0$. Let $E = \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{\infty} T^{-j}(T^i(a))$ be a chain-tree in D_2. We may then pick a chain $H = \{a_z\}_{z\in\mathbb{Z}}$ in E to be a set such that $T(a_z) = a_{z+1}$ and $a_0 = a$. We focus on the measure on H. First, we construct a set $\mathcal{A}_n$. Define $A_n = \{a_z\}_{z\in n\mathbb{Z}}$ to be every n-th node in the chain. We pick $\mathcal{A}_n = \bigcup_{i=1}^{\infty} T^{-in}(A_n)$. For each $\mathcal{A}_n$, the n different sets $\mathcal{A}_n, T^{-1}(\mathcal{A}_n), T^{-2}(\mathcal{A}_n), ...T^{-(n-1)}(\mathcal{A}_n)$ are disjoint, and $T^{-n}(\mathcal{A}_n) = \mathcal{A}_n$. These sets repeat as we take preimages. Therefore, if any two of these sets have different measures under μ, say $\mathcal{A}_n$ and $T^{-m}(\mathcal{A}_n)$, then the sub-sequences of $T^{-k}(\mathcal{A}_n)$ corresponding to these generated by the n^{th} and $n + m^{th}$ indices converge to different values and the proof is complete.

Next, assume that $\mu(\mathcal{A}_n) = \mu(T^{-1}(\mathcal{A}_n)) = ... = \mu(T^{-(n-1)}(\mathcal{A}_n))$ for all n. Let $\mu(E) = M$. Since $\mathcal{A}_n \cup T^{-1}(\mathcal{A}_n) \cup ... \cup T^{-(n-1)}(\mathcal{A}_n) = E$, we then have that $\mu(\mathcal{A}_n) = \frac{M}{n}$. Consider the set $B = \{a_z|\ |z| = 2^n \text{ for } n \in \mathbb{N}\}$. Then, there exists a subsequence of $\{T^{-i}(B)\}_{i\in\mathbb{N}}$ given by $\{T^{-i_k}(B)\}$ such that $T^{-i_k}(B)$ contains a_0 for each i_k. This shows that there exists a subsequence of the $\{T^{-i}(B)\}_{i\in\mathbb{N}}$ where the limit of the measures of the subsequence is positive.

However, consider that for any n, $B \subset \{a_z \,|\, z \in 2^n\mathbb{Z}\} \cup \{a_{-2^{n-1}}, a_{-2^{n-2}}, ..., a_{2^{n-1}}\}$. Note that since μ is a finite measure and E is an infinite subset of D_2, for any finite $S \subset E$, the preimages of S are disjoint and $\lim_{m\to\infty}\mu(T^{-m}(S)) = 0$. Take $S = \{a_{-2^{n-1}}, a_{-2^{n-2}}, ..., a_{2^{n-1}}\}$. Further, it is assumed that $\mu(\mathcal{A}_{2^n}) = \mu(T^{-1}(\mathcal{A}_{2^n})) = ... = \mu(T^{-(2^n-1)}(\mathcal{A}_{2^n}) = 2^{-n}M$, and so because $T^{-k}(\{a_z \mid z \in 2^n\mathbb{Z}\})$ is a subset of one of these 2^n sets,

$$\lim_{k\to\infty}\mu(T^{-k}(B)) \leq 2^{-n}M \tag{47}$$

Since we picked n arbitrarily, the limit then must be 0. The two subsequences converge to different values, giving a contradiction. □

The central part of the above argument is collapsing the structure of the chain-tree in such a way that its structure mirrors the integers. We may instead expand this to collapse a single level of the chain-tree to a set B_z corresponding to the element of the chain a_z. This more directly reduces considerations of the chain-tree to considerations of the chain (or the integers), where the Collatz map acts as a right shift. This argument allows us to extend the result further to the more-desirable Cesaro limits as opposed to standard limits, which presents them in a form more common to Ergodic Theory.

Definition 3. *Let $(\Omega, \mathcal{F}, \nu)$ be a probability measure space and $V : \Omega \to \Omega$ a $\mathcal{F}$-measurable map. Then, we say ν is asymptotically mean stationary with respect to V if $\lim_{N\to\infty}\dfrac{1}{N}\sum_{n=1}^{N}\nu(V^{-n}(A))$ exists for all $A \in \mathcal{F}$.*

Proposition 14. *Let there exists a finite measure μ on $\mathbb{N}$ such that $\lim_{k\to\infty}\dfrac{1}{k}\sum_{i=1}^{k}\mu(T^{-i}(A))$ exists for all $A \subset \mathbb{N}$, that is to say that $(\mathbb{N}, P(\mathbb{N}), \mu, T)$ is asymptotically mean stationary. Then, D_2 must have measure 0.*

Remark: Note that $\frac{1}{k}\sum_{i=1}^{k}\mu(T^{-i}(\mathcal{A}_n))$ converges to $\frac{1}{n}$ as $k \to \infty$. In other words, this is a much weaker requirement on the measure than the previous case. The limit does act similarly on finite sets. For any set A such that $\sum_{i=1}^{\infty}\mu(T^{-i}(A)) = l < \infty$, $\frac{1}{k}\sum_{i=1}^{k}\mu(T^{-i}(A)) \leq \frac{l}{k}$ shows that the limit of these means is 0. Since the above property holds for singletons, it holds for finite sets as well. The effort is then extending to the infinite case in the same way.

Proof. By contradiction, let μ be such a measure and $a \in D_2$ be a point so $\mu(a) > 0$. Let $H = \{a_z\}_{z\in Z}$ and E be as in the proof of proposition 1, and assume $\mu(E) = 1$ by renormalization.

We begin by redefining a set similar to the $\mathcal{A}_n$ in concept. Let $B_z = \bigcup_{i=0}^{\infty}T^{-i}(T^i(a_z))$, so that the B_z correspond to a "level" of the chain-tree E as demarcated by the a_z.

First, we construct a set B. To begin, select N such that $\sum_{i=N+1}^{\infty}\mu(B_i) + \mu(B_{-i}) < \frac{1}{20}$, so that also $\sum_{-N}^{N}\mu(B_i) \geq \frac{19}{20}$ (these values are mostly arbitrary choices, though the first must be sufficiently small for the following argument). Let $K = 2N+1$. Let

$$B = [B_{N+1} \cup B_{N+2} \cup ... \cup B_{3N+1}] \cup [B_{7N+4} \cup B_{7N+5} \cup ... \cup B_{11N+5}] \cup ... \cup [B_{19N+10} \cup B_{19N+11} \cup ... \cup B_{27N+13}] \cup ... \tag{48}$$

Considering levels as starting from level B_{-N-1}, we skip K levels, then, at each step, we take enough nodes so that the total number of levels included divided by the number of levels since $-N-1$ is $\frac{1}{2}$, then exclude enough that this drops to $\frac{1}{3}$, and exclude K more as a buffer. Looking at blocks of K nodes, where 1 represents inclusion and 0 exclusion, this looks like the sequence

$$0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, ... \tag{49}$$

This allows us to construct two sequences based on these choices which converge in different ways.

Consider first the value $\frac{1}{2K}\sum_{i=1}^{2K}\mu(T^{-i}(B))$. Since $T^{-1}(B_i) = B_{i-1}$, and since we move $2K$, we have that the K levels in B given by $B_{N+1}, ..., B_{3N+1}$ each take the values of $B_{-N}, ..., B_N$ precisely once in the sum $\sum_{i=1}^{2K}\mu(T^{-i}(B))$. Therefore,

$$\sum_{i=1}^{2K}\mu(T^{-i}(B)) \geq \left(\frac{19}{20}\right)\cdot K \tag{50}$$

Using similar equations, we may track preimages as they pass over the central mass of $\frac{19}{20}$ or stay within the tail mass of $\frac{1}{20}$ to construct two sequences with masses bounded as

$$\frac{1}{2K}\sum_{i=1}^{2K}\mu(T^{-i}(B)) \geq \frac{19}{20}\left(\frac{K}{2K}\right) = \frac{19}{40} \qquad \frac{1}{3K}\sum_{i=1}^{3K}\mu(T^{-i}(B)) \leq \frac{1}{20}+\frac{19}{20}\left(\frac{K}{3K}\right) \tag{51}$$

$$\frac{1}{6K}\sum_{i=1}^{6K}\mu(T^{-i}(B)) \geq \frac{19}{20}\left(\frac{3K}{6K}\right) = \frac{19}{40} \qquad \frac{1}{9K}\sum_{i=1}^{9K}\mu(T^{-i}(B)) \leq \frac{1}{20}+\frac{19}{20}\left(\frac{3K}{9K}\right) \tag{52}$$

$$\frac{1}{14K}\sum_{i=1}^{14K}\mu(T^{-i}(B)) \geq \frac{19}{20}\left(\frac{7K}{14K}\right) = \frac{19}{40} \qquad \frac{1}{21K}\sum_{i=1}^{21K}\mu(T^{-i}(B)) \leq \frac{1}{20}+\frac{19}{20}\left(\frac{7K}{21K}\right) \tag{53}$$

For the pattern on the left, we consider indices such that half of the constructed sequence up to that point (considered from $-N-1$) is included in B. In $2k$, there are k left out and k included. Similarly, in $6K$, there are k out, k in, $2k$ out, $2k$ in, leaving $3k$ in and $3k$ out. The measures of these are always at least $\frac{19}{40}$ by the same computation as for the cases shown above. The pattern on the right corresponds to going far enough forward that $1/3$ of levels since $-N-1$ are included and $2/3$ excluded. The construction of B guarantees that the corresponding values are at most $\frac{1}{20}+\frac{19}{60}$. This constructs two subsequences which must have different limits and contradicts the assumption that the limit converges. □

Remark:

i) Constructing a measure with either the property in proposition 1 or proposition 2 which is only zero on points known to be in $C \cup D_1$ then this shows that D_2 is empty as well.
ii) This measure has weaker requirements than that posed in [2]
iii) This proposition does not require that T be nonsingular, as the following proposition does.

A slight modification of this proposition allows for an argument based on Birkhoff's Ergodic Theorem which generalizes to the class of maps posed in [3] as well as to more general measurable maps. The second version also relies on one extra proposition due to Gray and Kieffer [6]. The proof for Proposition 7 given below may be found in U.Krengel's book [10]. First, recall that for a probability measure space $(\Omega, \mathcal{F}, \nu)$, a measurable map $V : \Omega \to \Omega$ is null-preserving or nonsingular of $\nu(A) = 0$ implies $\nu(T^{-1}(A)) = 0$.

Proposition 15. *The system $(\Omega, \mathcal{F}, \nu, V)$, for V nonsingular, is asymptotically mean stationary if and only if the averages $\frac{1}{N}\sum_{n=1}^{N} f(V^n(x))$ converges ν-almost everywhere for each bounded, measurable function f.*

Proof. For the reverse direction, note that $\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_A(T^n(x))$ converges for each indicator function $\mathbb{1}_A$ and $A \in \mathcal{F}$. Thus,

$$\int_\Omega \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_A(T^n(x)) = \frac{1}{N}\sum_{n=1}^{N}\nu(V^{-n}(A)) \tag{54}$$

converges by the Dominated convergence theorem.

For the forward direction, assume that ν is a probability measure. Then, by assumption, $\bar{\nu}(A) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\nu(V^{-n}(A))$ defines a measure by the Vitali-Hahn-Saks Theorem ($\nu_N(A) = \frac{1}{N}\sum_{n=1}^{N}\nu(V^{-n}(A))$ for all measurable A are absolutely continuous with respect to ν and converge).

Consider the set B_f of points ω for which $\frac{1}{N}\sum_{n=1}^{N} f(V^n(x))$ converges. The set B_f is clearly V-invariant and so $\nu(B_f) = \bar{\nu}(B_f)$, where V is measure-preserving with respect to $\bar{\nu}$, so that the Birkhoff-Khinchin theorem gives $\bar{\nu}(B_f) = 1$. □

Proposition 16. *Let there exists a finite measure μ on $\mathbb{N}$ everywhere nonzero such that $(\mathbb{N}, P(\mathbb{N}), \mu, T)$ is asymptotically mean stationary. Then, $D_2 = \emptyset$.*

Remark: Note that T is nonsingular with respect to μ in this case.

Proof. The Hopf decomposition in [2] and [3] shows that the set D_2 is an at-most countable union of wandering sets, or

$$D_2 = \bigcup_{n=0}^{\infty} W_n \tag{55}$$

where $\mu(T^i(W_n) \cap T^j(W_n)) = \emptyset$ for $i \neq j$.

Let m be the restriction to D_2 of the limiting measure $\bar{\mu}$ in the proof of the above proposition. Note that m is T-invariant. If $\mu(D_2) > 0$, then $m(D_2) > 0$, and in particular, $m(W_n) > 0$ for a wandering set W_n. Then, $m(\bigcup_{j=0}^{\infty} T^{-j}(W_n)) \leq m(D_2) < \infty$, but $m(\bigcup_{j=0}^{\infty} T^{-j}(W_n)) = \sum_{j=1}^{\infty} m(W_n) = \infty$, a contradiction. $\square$

This direction of the argument, that the existence of a measure implies something about D_2, is the more important direction because it allows for statements on the Collatz map. However, the converse is also true. The following argument immediately generalizes from the Collatz map to any Syracuse-type map such that the preimage of a finite set is finite.

Lemma 17. *If D_2 is empty, there exists an everywhere-nonzero finite measure μ asymptotically mean stationary with respect to the Collatz map.*

Proof. Recall the decomposition $N = C \cup D_1 \cup D_2$ (where we assume $D_2 = \emptyset$ here). Let $C = \bigcup_{i=1}^{\infty} C_i$ where fore each i, $C_i = \{c_1, .., c_{n_i}\}$ is a cycle.

Then we construct μ to be a probability measure. Set $\mu(c_1) = ... = \mu(c_{n_i}) = \frac{1}{2^{i+2} n_i}$. Then, for $k \geq 0$, we take the $\mu(T^{-k}(T^{-1}(c_j) \backslash C_i)) = \frac{1}{2}\mu(T^{1-k}(T^{-1}(c_j) \backslash C_i))$ where all values in this set have equal measure. This is to say that we consider the branch of D_1 mapping to each c_j in this cycle (without intersecting the cycle elsewhere before mapping to c_j) and weight these equally across the c_j values. It is immediate that $\mu(\mathbb{N}) = \sum_{i=1}^{\infty} 2\mu(C_i) = 1$.

Consider first $A \subset D_1$. Then, $\sum_{j=0}^{\infty} \mu(T^{-j}(A)) < \infty$ implies

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} \mu(T^{-j}(A)) = 0 \tag{56}$$

Next, assume that A is a subset of a single cycle $C_i = \{c_1, ..., c_{n_i}\}$. We have that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} \mu(T^{-j}(A \cap C_i)) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} \mu(T^{-j}(\{c_{j_1}, ..., c_{j_l}\})) \tag{57}$$

$$= \lim_{N \to \infty} \frac{l}{N} \left(\sum_{m=0}^{N} \frac{2^{-m}(N-m)}{2^{i+2} n_i} \right) = l\left(\sum_{m=0}^{\infty} \frac{2^{-m}}{2^{i+2} n_i}\right) = \frac{l}{2^{i+1} n_i} \tag{58}$$

The argument immediately generalizes to any subset of C with a bit more algebra in computing the actual limit. Therefore, since any subset of the natural numbers may be decomposed $A = (A \cap C) \cup (A \cap D_1)$, μ is asymptotically mean stationary with repect to T. $\square$

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